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Semiclassical approach to time-dependent tunnelling

J.C. Martinez[a], E. Polatdemir

Natural Sciences Group, National Institute of Education, # 7-03-110, Nanyang Technological University, 1 Nanyang Walk, Singapore 637616

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Abstract. We present an exact time-dependent semiclassical formulation of the dynamics of a δ-atom subjected to an oscillating electric field. Through a simple approximation the results of Ergenzinger's more intuitive analysis are obtained. We also comment on the important role played by the imaginary tunnelling time t_0, which is quite distinct from the usual adiabatic tunnelling time.

1 Introduction

Whenever a quantum system is amenable to a semiclassical treatment, one is aware that a dividend is soon to be declared: despite the fact that the semiclassical approximation holds for small $\hbar$, it is not a perturbative expansion and hence it is not usually plagued by difficulties that accompany perturbation theories. The semiclassical approach has been applied successfully to many systems, notably those exhibiting the tunnelling phenomenon in its various manifestations [1]. For convenience one can think of the states of the system as almost stationary states since the tunnelling probability is small enough to be ignored and the particle flux current is essentially constant [2]. Given this state of affairs, one carries out the calculation of transition amplitudes in energy space. There is, however, a dearth of applications of the method to systems that are truly time dependent and this paucity of model problems is now proving to be serious [3]. Because the energy is no longer fixed, even approximately, in this second case, one cannot go into energy space in a simple way. This exposes a lurking anxiety about the semiclassical method: although the tunnelling phenomenon may be calculable, this does not mean that we fully comprehend it [3,4]. In other words, the time-dependent case demands greater understanding of the tunnelling phenomenon. Admittedly, the issue in question harks back, in part, at the foundations of quantum theory itself and may involve more than just time dependence. Nevertheless, one expects that useful, if modest, insights into the tunnelling problem can be gained by examining tractable time-dependent problems.

Not long ago, Ergenzinger [5] applied the semiclassical method to the ionization of a one-dimensional δ-atom by an oscillating electric field. The interplay between the field and the attractive δ-potential has been a natural complication hampering a complete analytical solution. Ergenzinger's approach of using time-averaged phases and leaning upon an intuitive application of the time-independent theory to a time-dependent system, proved highly successful in reproducing predictions obtained previously from numerical simulations [6]. Nevertheless, in light of the remarks of the foregoing paragraph, the issue of time dependence was not sufficiently appreciated, nor was his formulation based on a full development of the machinery available for semiclassical calculations. In this paper, we wish to supply the justification for Ergenzinger's method and elucidate some details about tunnelling from a time-dependent setting. In fact, within the framework of the semiclassical method, our formulation is exact. While physically tunnelling is always a time-dependent process, the formalism in current use conveniently looks upon it within the framework of stationary-state problems and, in so doing, demystifies the tunnelling process to the point of rendering further inquiry into the phenomenon unnecessary [3]. As this seems to be no longer the case in a time dependent setting, one must now grapple with the issue squarely. Moreover, in recent years the theme of a quantum tunnelling time has become fashionable once more and the new development that Ergenzinger's work brings with it is that, unlike most of the model systems being studied in this connection, his problem is an almost exactly solvable time-dependent one. As we will see below an imaginary tunnelling time will be required to reproduce results predicted previously by simulation calculations. We might mention that a study of time-dependent tunnelling problems to which standard methods cannot be applied straightforwardly had been made not long ago [7].

The physical system under study has a long history, that we cannot go into in detail here [8]. Because our main focus is in providing insight into the semiclassical approach applied to a time-dependent problem, it will suffice to observe that the ionization of the δ-atom continues to be a focal point of current theoretical and experimental research [5,6]. The system is a one-dimensional δ-potential atom subjected to an external oscillating electric field. In

[a] e-mail: jcmartin@nie.edu.sg

the dipole approximation the relevant Schrodinger equation is

$$\mathrm{i}\hbar\frac{\partial}{\partial\tilde{t}}\Psi(\tilde{x},\tilde{t}) = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial\tilde{x}^2} - \alpha\delta(\tilde{x}) - \mu\tilde{x}\cos\omega\tilde{t}\right)\Psi(\tilde{x},\tilde{t}), \tag{1}$$

in which it will be convenient to represent distance and time by $\tilde{x}$ and $\tilde{t}$, respectively. With the external field switched off, the atom has a single bound state [9] with the ground-state energy, $E_0 = -m\alpha^2/2\hbar^2$, and a normalized time-independent wave function, $\Psi_0(\tilde{x}) = (m\alpha/\hbar)^{1/2}\exp(-m\alpha|\tilde{x}|/\hbar)$. Hence unlike the usual case of tunnelling from an initial positive-energy state, the present problem deals with an initial bound state. Three physical dimensionless parameters can be identified. There is, first, the ratio of the ponderomotive energy $\mu^2/4m\omega$ to the photon energy $\hbar\omega$, $z = \mu^2/4m\hbar\omega^3$. This is a measure of the field strength and we expect spikes in the ionization rate at intervals of $\Delta z = 1$. The second is the Keldish factor, which is the ratio of the adiabatic tunnelling time to the period of the electric field: $\gamma = \alpha\omega/\hbar\mu$ [10]. One is interested in a Keldish factor of order unity, because approximation methods are known to fail in that regime, while the semiclassical theory should do well. Small γ ($<< 1$) corresponds to tunnelling; large-γ processes are best treated within a multiphoton description and will not be of interest here [11]. Finally, the third, though not independent, parameter is the ratio of the binding energy to the photon energy, $n_{io} = 2\gamma^2 z$.

From an examination of the interplay of these parameters within the context of the δ-atom, we can expect an imaginary time to play an important role. Suppose we keep γ and ω constant while allowing the field strength and binding energy to vary. Increasing the field strength gives rise to a quadratic growth in z; and n_{io} increases linearly with z. Hence, an increase in the field strength implies a correspondingly much larger rise in the binding energy and this means that the ionization rate would decrease with increase in z. This decay is exponential in z because the rate depends on the number of available ionizing atoms. It follows that the decay constant associated with z does not depend on z or n_{io} and hence must be a function of γ. Thus we expect an imaginary time $t_0 = t(\gamma)$ to be associated with z in the expression for the ionization rate. Naturally, we surmise that this time is a measure of the tunnelling time of the system. Of course, the next logical question is: is the tunnelling time real and measurable? Without commenting on the latter issue, we will argue below that an imaginary tunnelling time is real and meaningful in our context.

The rest of the paper is organized as follows. Section 2 is devoted to the calculation of the propagator for the δ-atom. We first employ the semiclassical formula to obtain the exact propagator of an electron is an oscillating field, with the δ-interaction switched off. Then we carry out a perturbation expansion of the full propagator in terms of the δ-interaction. Next, in Sect. 3, the tunnelling process is incorporated into the propagator obtained in the previous section and an expression for the probability amplitude that the atomic electron is not ionized is given. Then, in Sect. 4, we comment on the imaginary tunnelling time described in the previous paragraph and defined in Sect. 3. Finally, we sum up our conclusions in Sect. 5.

2 Semiclassical propagator

This section is devoted to the derivation of the propagator for the δ-function atom subject to an external electric field. Following Ergenzinger [5] we define new variables

$$h = \frac{\hbar\omega^3}{\mu^2}, \quad \gamma = \frac{\alpha\omega}{\hbar\mu}, \quad \tilde{t} = \omega t, \quad \tilde{x} = \frac{\omega^2}{\mu}x, \tag{2}$$

and recast (1) in the form

$$\mathrm{i}h\frac{\partial}{\partial t}\Psi = \left(-\frac{h^2}{2m}\frac{\partial^2}{\partial x^2} - \gamma h\delta(x) - x\cos t\right)\Psi. \tag{3}$$

This equation resembles a standard quantum mechanical problem in which h takes the place of Planck's constant $\hbar$. Therefore, the usual semiclassical method is applicable in the limit of small h. Since $h = 1/(4mz)$ is essentially the ratio of photon energy to ponderomotive energy (for unit mass) this limit corresponds to very intense electric fields.

As a first step toward obtaining the required propagator, let us apply the semiclassical method to the simpler problem of a free charged particle in the presence of an oscillating electric field. In the dipole approximation the Lagrangian of this problem is

$$L_0(x,\dot{x},t) = \frac{1}{2}m\dot{x}^2 + x\cos t. \tag{4}$$

From the equation of motion, $m\ddot{x} - \cos t = 0$, we may write the classical action as

$$S_0[x_{\mathrm{cl}}(t)] = \frac{1}{2}mx\dot{x}\Big|_{t'}^{t''} + \frac{1}{2}\int_{t'}^{t''} x\cos t\mathrm{d}t, \tag{5}$$

where the end points are $x(t') = x'$ and $x(t'') = x''$. The classical path x_{cl} for this system is just

$$\begin{aligned} mx_{\mathrm{cl}}(t) = &-\cos t + \cos t' + mx' \\ &+\frac{mx'' - mx' + \cos t'' - \cos t'}{t'' - t'}(t - t'), \end{aligned} \tag{6}$$

whose corresponding action is

$$\begin{aligned} S_0[x_{\mathrm{cl}}] = S_0[x'',t'';x',t'] = \frac{1}{2}m\Big\{ & x''\sin t'' - x'\sin t' \\ &+\frac{mx'' - mx' + \cos t'' - \cos t'}{t'' - t'}(x'' - x')\Big\} \\ &+\frac{1}{2m}\Big\{ -\frac{1}{2}(t'' - t') - \frac{1}{4}(\sin 2t'' - \sin 2t') \\ &+(\cos t' + mx')(\sin t'' - \sin t') \\ &+\frac{mx'' - mx' + \cos t'' - \cos t'}{t'' - t'} \\ &\times((t'' - t')\sin t'' + \cos t'' - \cos t')\Big\}. \end{aligned} \tag{7}$$

It will also be convenient to have an expression for the time difference $t''-t'$,

$$t''-t'=\frac{x''-x'}{v_0}+\frac{\cos t''-\cos t'}{mv_0}, \tag{8}$$

where we defined the momentum $mv_0=(mx''-mx'+\cos t''-\cos t')/(t''-t')$.

We calculate the exact propagator from the semiclassical formula (note that $\hbar$ is replaced by h) [2,12]

$$\begin{aligned}K_0(x'',t'';x',t')&=\sqrt{\frac{1}{2\pi \mathrm{i}h}\left|\frac{\partial^2 S_0}{\partial x'\partial x''}\right|}\mathrm{e}^{(\mathrm{i}/h)S_0[x'',t'';x',t']}\\&=\sqrt{\frac{m}{2\pi \mathrm{i}h(t''-t')}}\mathrm{e}^{(\mathrm{i}/h)S_0[x'',t'';x',t']}.\end{aligned} \tag{9}$$

That this is the exact propagator for L_0 can be verified by expanding an arbitrary path $x(t)$ connecting the end points (x'',t'') and (x',t') in terms of the classical path: $x(t)=x_{\mathrm{cl}}(t)+\delta x(t)$. After an integration by parts the classical action is written as a sum of two terms,

$$S_0[x]=S_0[x_{\mathrm{cl}}]+\int_{t'}^{t''}\frac{1}{2}m(\delta\dot{x})^2\mathrm{d}t, \tag{10}$$

with the understanding that $\delta x(t')=\delta x(t'')=0$, by virtue of the end point conditions. It follows that the propagator is $\exp(\mathrm{i}/h)S_0(x_{\mathrm{cl}})$ multiplied by the propagator for a particle evolving freely from the origin back to the origin [13]. This is just what we have above. The exactness of the semiclassical method in this case comes as no surprise, because the semiclassical method is known to yield the correct result for potentials that are at most quadratic in x, *even* when the coefficient is time dependent [14].

The Lagrangian for the original system described by (3),

$$L(x,\dot{x},t)=\frac{1}{2}m\dot{x}^2+x\cos t-a\delta(x), \tag{11}$$

where $a=h\gamma$, can be addressed by following the method of Lawande and Bhagwat [15]. We expand the propagator for this system by treating the δ-potential as a perturbation. Thus we write formally

$$\begin{aligned}&K(x'',t'';x',t')\\&\quad=\int D[x(t)]\mathrm{e}^{(\mathrm{i}/h)(S_0[x(t)]+S_1[x(t)])}\\&\quad=\int D[x(t)]\mathrm{e}^{(\mathrm{i}/h)S_0[x]}\left(1+\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{\mathrm{i}}{h}\right)^n(S_1)^n\right)\\&\quad\equiv K_0(x'',t'';x',t')+K_1(x'',t'';x',t').\end{aligned} \tag{12}$$

In the above, the path differential measure is denoted by $D[x(t)]$ and K_0 is the propagator for the unperturbed system given by (9). Equation (12) also serves as the definition of K_1, the contribution of the δ-potential to the full propagator.

To proceed efficiently with the rest of the calculation, it is convenient to single out the nth term of the sum,

$$G_n(x'',t'';x',t')=\frac{1}{n!}\int D[x(t)]\mathrm{e}^{(\mathrm{i}/h)S_0[x]}(S_1)^n, \tag{13}$$

$$K_1(x'',t'';x',t')=\sum_{n=1}^{\infty}\left(\frac{\mathrm{i}}{h}\right)^n G_n(x'',t'';x',t'). \tag{14}$$

By virtue of the δ-interaction, it follows that

$$(S_1)^n=(-a)^n n!\int_{t'}^{t''}\mathrm{d}t_n\int_{t'}^{t_n}\mathrm{d}t_{n-1}\ldots\int_{t'}^{t_2}\mathrm{d}t_1\prod_{i=1}^{n}\delta(x_i), \tag{15}$$

where the time-order sequence $t'\le t_1\le t_2\le\ldots\le t_n\le t''$ holds; finally,

$$\begin{aligned}&G_n(x'',t'';x',t')\\&\quad=(-a)^n\int_{t'}^{t''}\mathrm{d}t_n\int_{t'}^{t_n}\mathrm{d}t_{n-1}\ldots\int_{t'}^{t_2}\mathrm{d}t_1K_0(x'',t'';0,t')\\&\qquad\times\prod_{k=2}^{n}K_0(0,t_k;0,t_{k-1})K_0(0,t_1;x',t').\end{aligned} \tag{16}$$

To make further advance, define the Laplace transform of G_n:

$$\tilde{G}_n(x'',s;x_n,0)\equiv\int_0^{\infty}\mathrm{e}^{-st}\mathrm{d}tG_n(x'',t;x_n,0). \tag{17}$$

From the theory of Laplace transforms [16] the convolution of two functions $f(t)$ and $g(t)$ is defined by

$$(f\circ g)(t)\equiv\int_{-\infty}^{\infty}f(x)g(t-x)=(g\circ f)(t), \tag{18}$$

and the convolution theorem states that

$$\int_0^{\infty}\mathrm{e}^{-st}(f\circ g)(t)\mathrm{d}t=\int_0^{\infty}\mathrm{e}^{-st}f(t)\mathrm{d}t\int_0^{\infty}\mathrm{e}^{-sv}g(v)\mathrm{d}v. \tag{19}$$

We will apply these results to the evaluation of $G_n(x'',t'';x',t')$. We go back to (9) and approximate $K_0(x'',t'';x',t')$ by discarding all the oscillatory terms in the exponent; in other words, replacing them by their time averages. For instance we have

$$\begin{aligned}K_0(x'',t'';0,t_n)\approx\sqrt{\frac{m}{2\pi \mathrm{i}h(t''-t_n)}}\exp\left\{\frac{\mathrm{i}m}{2h(t''-t_n)}x''^2\right.\\\left.-\frac{\mathrm{i}}{4hm}(t''-t_n)\right\}.\end{aligned} \tag{20}$$

This step is equivalent to Ergenzinger's procedure of replacing the phase in the transition amplitude by its time-averaged value [5]. In our case this step is taken so the

conditions of the convolution theorem apply. This is the only approximation in the paper; although an exact evaluation is possible that would require numerical techniques. It now follows from (16) that

$$\tilde{G}_n(x'',s;x_n,0) \approx (-a)^n \tilde{K}_0(x'',s;0,0) \times[\tilde{K}_0(0,s;0,0)]^{n-1}\tilde{K}_0(0,s;x',0), \quad (21)$$

in which the Laplace transform of, e.g., $K_0(x'',t;0,0)$ is

$$\begin{aligned}\tilde{K}_0(x'',s;0,0) &\approx \int_0^\infty \mathrm{e}^{-st}\sqrt{\frac{m}{2\pi \mathrm{i}ht}}\exp\left(\frac{\mathrm{i}m}{2ht}x''^2 - \frac{\mathrm{i}}{4hm}t\right)\mathrm{d}t \\ &= \sqrt{\frac{m}{2\mathrm{i}h\varsigma}}\exp\left(-2\sqrt{-\varsigma\frac{\mathrm{i}mx''^2}{2h}}\right), \end{aligned} \quad (22)$$

where $\varsigma \equiv s + \mathrm{i}/4hm$. The symbol $\approx$ is used in various places as a reminder of our simplification of the phase by ignoring the oscillatory terms.

Referring to (14), we take the inverse Laplace transform,

$$\begin{aligned}K_1(x'',t'';x',t') &= -\frac{\mathrm{i}am}{2\mathrm{i}h^2}\frac{1}{2\pi i}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty}\mathrm{e}^{st}\mathrm{d}s\frac{1}{\varsigma^{1/2}} \\ &\times\frac{1}{\varsigma^{1/2}+a\left(\frac{\mathrm{i}m}{2h^3}\right)^{1/2}}\exp\left(-2\xi\sqrt{-\frac{\mathrm{i}m}{2h}\varsigma}\right), \end{aligned} \quad (23)$$

in which $\xi \equiv |x'| + |x''|$. Let $\varsigma = \omega^2$, use an integral representation of $\{\omega + a(\mathrm{i}m/2h^3)^{1/2}\}^{-1}$; define also the half-circle contour C in the complex t-plane, which is a line parallel to the imaginary axis and closed by a circular arc, to obtain ($T = t'' - t'$):

$$\begin{aligned}K_1(x'',t'';x',t') &= -\frac{am}{2\pi i h^2}\mathrm{e}^{-\mathrm{i}T/4hm}\int_C \mathrm{d}\omega\int_0^\infty \mathrm{d}\zeta \mathrm{e}^{T\omega^2 - 2\xi\omega(-\mathrm{i}m/2h)^{1/2}} \\ &\times\mathrm{e}^{-\zeta\left(\omega+a(\mathrm{i}m/2h^3)\right)}. \end{aligned} \quad (24)$$

We deform the contour C to coincide with the imaginary axis, and after some manipulations, we find

$$K_1(x'',t'';x',t') = \frac{am}{h^2}\mathrm{e}^{-\mathrm{i}T/4hm}\int_0^\infty \mathrm{d}sG(s+\xi,T;0,0) \times\mathrm{e}^{-ams/h^2}, \quad (25)$$

where the free particle propagator in our variables is given by

$$G(s,T;0,0) = \sqrt{\frac{m}{2\pi\mathrm{i}hT}}\exp\left\{-\frac{m}{2\mathrm{i}hT}s^2\right\}. \quad (26)$$

Looking back at the derivation of (25), we recall that the phase factor $\mathrm{e}^{-\mathrm{i}T/4hm}$ (where $T = t'' - t'$) arose from the non-zero times in $K_0(x'',t'';0,t_n)$ [see (20)]. However, in the next section, we will be interested only in the case that x' and x'' coincide with the origin and for full cycles of the electric field, that is, for $t'' = 2\pi n$ and $t' = 0$. Referring to (8), this means that $t''-t'$ may be effectively replaced by zero. Henceforth, we will not carry this factor. An identical factor in $\mathrm{e}^{\mathrm{i}S_0/h}$ can also be removed from $K_0(x'',t'';x',t')$. So, putting all the elements together, we now have the required propagator,

$$\begin{aligned}K(x'',t'';x',t') &\approx \sqrt{\frac{m}{2\pi\mathrm{i}h(t''-t')}}\mathrm{e}^{(\mathrm{i}/h)S_0[x'',t'';x',t']} \\ &-\frac{1}{2}\frac{m|\gamma|}{h}\mathrm{e}^{-m|\gamma|\xi/h}\mathrm{e}^{-m\gamma^2(t''-t')/2\mathrm{i}h}. \end{aligned} \quad (27)$$

Note that $\gamma < 0$. From the preceding discussion, we stipulate that S_0 is given by (7) but *without* the term linear in $t'' - t'$. Also in the limit that $h \to 0$, we have $(1/2)(m|\gamma|/h)\mathrm{e}^{-m|\gamma|\xi/h} \to \delta(\xi)$, so the second term may be replaced by [5]

$$-\delta(\xi)\mathrm{e}^{\mathrm{i}m\gamma^2(t''-t')/2h}, \quad (28)$$

that is, for small h, there is a strong localization of the wave function around the origin. The importance of localization properties had been previously reviewed by Casati and Molinari [17].

3 Tunnelling propagator

The goal of this section is the inclusion of a single tunnelling event in the propagator obtained above. The atomic electron, initially at position $(y, t' = 0)$, propagates to the final position (x, t'') while undergoing a tunnelling event in between, at $(z, t = t_0)$. It will be convenient to choose complete cycles so that $t'' = 2n\pi$. The bound state corresponding to the delta potential; $\gamma h\delta(x)$ in (3) has the negative energy $E_0 = -m\gamma^2/2$ which corresponds to a complex momentum of $m\dot{x} = \pm\mathrm{i}m\gamma$ [9]. For simplicity, we assume that at the time t_0 the electron has just this complex momentum [5]. To ensure the exponential decay of the wave functions, let us pick

$$\dot{x}(t = t_0) = +\mathrm{i}\gamma. \quad (29)$$

The time t_0 is imaginary,

$$t_0 = \mathrm{i}\ln\left(m\gamma + \sqrt{(m\gamma)^2+1}\right) \quad (30)$$

and $\sin t_0 = \mathrm{i}m\gamma$ and $\cos t_0 = (1 + (m\gamma)^2)^{1/2}$. However, the time t_0 is not unique, for tunnelling could also occur at times $t_0 + \pi k, k = 0, 1, 2, 3, \ldots$ Because of the oscillations of the electric field, these events are interpreted as the tunnelling of the electron to the right from the origin [for $t_0 + 2k\pi$] or to the left [for $t_0 + (2k+1)\pi$]. Then using the

composition rule for propagators and (27) and (28), the required result is

$$\begin{aligned}
&K(x,t'';y,t'=0)\\
&=\int \mathrm{d}z \sum_k K(x,t'';z,t_0+\pi k)K(z,t_0+\pi k;0,t')\\
&=\int \mathrm{d}z\Bigg\{\sum_k (K_0(x,t'';z,t_0+\pi k)K_0(z,t_0+\pi k;0,t')\\
&\quad -K_0(x,t'';z,t_0+\pi k)\delta(|y|+|z|)\mathrm{e}^{-\mathrm{i}(m\gamma^2/2h)(t''-t_0-\pi k)}\\
&\quad -K_0(z,t_0+\pi k;y,0)\delta(|x|+|z|)\mathrm{e}^{-\mathrm{i}(m\gamma^2/2h)(t_0+\pi k)})\\
&\quad +\delta(|x|+|y|)\delta(|y|+|z|)\mathrm{e}^{-\mathrm{i}(m\gamma^2/2h)t''}\Bigg\}, \qquad (31)
\end{aligned}$$

where $k=0,1,2,3,\ldots$ such that $0\le \pi k\le t''$.

The effect of the electric field on the atomic electron is that of rendering the bound state metastable through tunnelling. In a static picture of the situation, the bound state has an exponentially decaying probability of tunnelling through the potential barrier leaving the atom ionized. This can be accounted for phenomenologically by adding to the ground energy E_0 a negative imaginary part. Following Ergenzinger [5] this additional term is taken to be the time average of $h/2$ times the decay rate calculated for a static barrier, γx using the standard WKB formula. Thus we make the replacement

$$E_0=\frac{m\gamma^2}{2h}\to\frac{m\gamma^2}{2h}-\mathrm{i}\frac{\gamma^2}{2}\sqrt{\frac{3h}{\pi\gamma^3}}\mathrm{e}^{-2m^2\lambda^3/3h}\equiv\bar{E}_0, \qquad (32)$$

to the ground-state energy. The evaluation of (31) now proceeds straightforwardly,

$$\begin{aligned}
K(x,t'';y,0)=&\sqrt{\frac{m}{2\pi\mathrm{i}ht''}}\\
&\times\mathrm{e}^{(\mathrm{i}m)/(2ht'')[x-y+\frac{1}{2m}(\cos t''-1)]^2}\\
&\times\mathrm{e}^{(\mathrm{i})/(4hm)(\sin 2t''+2mx\sin t'')}\\
&+\delta(|x|-|y|)\mathrm{e}^{-\mathrm{i}t''\bar{E}_0/h}\\
&-\sum_k \mathrm{e}^{-\mathrm{i}(t''-t_0-\pi k)\bar{E}_0/h}\sqrt{\frac{m}{2\pi ih(t_0+\pi k)}}\\
&\times\mathrm{e}^{(\mathrm{i}/h)S_0[x,t_0+\pi k;y,0]}\\
&-\sum_k \mathrm{e}^{-\mathrm{i}(t_0+\pi k)\bar{E}_0/h}\sqrt{\frac{m}{2\pi\mathrm{i}h(t''-t_0-\pi k)}}\\
&\times\mathrm{e}^{(\mathrm{i}/h)S_0[x,t'';y,t_0+\pi k]}, \qquad (33)
\end{aligned}$$

where the sum is from $k=0$ to $2n-1$ and the action terms S_0 are calculated from (7) without the term linear in the time difference. This is then the full propagator including tunnelling effects.

The first term of (33) simply describes the propagation of a negative energy electron (in an oscillating field) without ever coming in contact with the δ-potential at the origin. This term will not be of interest in connection with ionization. The next term is the background term: the electron is confined to the origin and evolves temporally in that position. There the field, being zero, has no effect on the electron. The third term stands for the electron interacting only with the field until $t=\pi k$; then it tunnels into the atom from this time to a time $t=\pi k+t_0$ and thereafter remains in the atom until the final time t''. Clearly this is a 'non-ionizing' event. Finally, the last term describes the particle remaining bound to the atom from $t'=0$ to $t=\pi k$; then it tunnels out from this time to $t=\pi k+t_0$ and continues propagating, subject only to the electric field, until the final time t''. Thus there are four possible outcomes and the scenario Ergenzinger was looking into is given by the second and fourth terms. The sum over k encapsulates the interference phenomenon of all the wave packets tunnelling at different times.

The wave function at the time t'' is calculated by applying the propagator to the ground state,

$$\Psi(x'',t'')=\int \mathrm{d}x' K(x'',t'';x',t')\Psi_0(x'), \qquad (34)$$

where only the second and last terms of K contribute to ionization. The probability amplitude p for the electron remaining bound is the projection onto the ground state,

$$p=\int \mathrm{d}x''\Psi(x'',t'')\Psi_0(x''). \qquad (35)$$

In both equations we may use for the ground-state wave function the expression $2(h/m\gamma)^{1/2}\delta(x)$, which holds for small h. This effectively forces x' and x'' to coincide with the origin. Recall also that we set $t''=2\pi n$, and $t'=0$ [cf. remarks at the end of Sect. 3]. Our results are identical to Ergenzinger's and we refer to his paper for graphs of the ionization rate [5].

4 Discussion

We discuss here some salient points about the imaginary time t_0 introduced in (30). The atomic electron propagates in complex time: under the electric field it evolves in real time whereas tunnelling takes place in the imaginary time between t_0 and the final time. This is essentially the Buettiker–Landauer idea which associates tunnelling with a process evolving in imaginary time while real-time propagation occurs in the perpendicular (independent) degrees of freedom [18]. Unfortunately, at present there is no unified consensus about the tunnelling (also traversal and sojourn) time [19]; in fact, there are several competing and even contradictory ideas. However, it seems clear that an imaginary time τ_0 can be usefully associated with the tunnelling process and we may take for it an expression proposed by Ivlev and Mel'nikov [8], namely

$$\tau_0=-\mathrm{Im}\frac{\partial}{\partial E}S(x'',x';E), \qquad (36)$$

in which S is the action in energy space. To estimate τ_0, we take for S the exponent of (22), where $s=E/h$. Setting $x''=|E_0|$, $x'=0$, we obtain

$$\tau_0=\frac{\mathrm{i}}{2}m\gamma, \qquad (37)$$

which, for small $m\gamma$, is of the same order of magnitude as t_0. We will point out below, however, that t_0 is quite distinct from the adiabatic tunnelling time.

Although the complex time coordinate is a useful device for describing tunnelling processes, it is really crucial in the present instance, since t_0 appears in the ionization rate [cf. (33)–(35)] and plays an important role in fitting the rate to the results obtained from numerical simulations. Actually it is much more important than the decay time [which can be extracted from (32)] whose inclusion alters the results imperceptibly. In (32), we introduced a complex energy whose imaginary part had the task of mimicking the decay in time of the ionization rate due to tunnelling. Similarly the time t_0 mimics the decay of the ionization rate as a function of the electric field energy. This ionization rate as a function of field energy is not a simple exponential owing to the delicate interference effects arising from the sum over k in the propagator.

Suppose that the field oscillates slowly so we have an almost static case. If the field is doubled, z is quadrupled. We can still keep γ constant by also doubling the binding strength of the atom. Then the tunnelling time is unchanged and the ionization rate remains the same. But in reality the ionization rate decays exponentially with z (for fixed γ) as noted in the Introduction and this is borne out by the graphs of Ergenzinger [5]. The quantity t_0 is a measure of this decay and must be related with the truly time-dependent dynamics of the system. From this we can see some features distinguishing the adiabatic tunnelling time from t_0. The former is a measure of the duration required by a lower-energy particle to penetrate a potential barrier; it is also related to the depletion of the number of particles available for tunnelling in time. The imaginary time t_0 is connected with the interplay between atomic binding and the electric field in which binding takes the dominating role; it also measures the increasing tendency of the atom to remain intact as the binding energy and the field strength grow in pace. It is clear that a deeper analysis of t_0 is required and that the semiclassical approach is well poised to address this issue.

A puzzling point might be why the bound-state energy for the field-free case was used in the calculation for the time t_0 in (30) for the full atom-in-field problem. Suppose we solve (3) approximately by replacing any $\sin^2 t$ by its time average, i.e., 1/4. Then an approximate bound solution is

$$\Psi \approx \mathrm{e}^{-m\gamma|x|}\mathrm{e}^{\left(-h/(2\mathrm{i})m\gamma^2+1/(4\mathrm{i}hm)\right)t} \times \mathrm{e}^{-(x/\mathrm{i}h)\sin t}\mathrm{e}^{\gamma\cos t}. \tag{38}$$

We now take the time average of the phase and calculate the corresponding wave function in momentum space to find the poles occurring exactly at the same place where they do for the field-free case. Therefore, when we employ a time-averaged phase (consistent with the approximation used above), the momentum corresponding to the bound-state has the same numerical value for the field-free and non-zero field cases.

Earlier, we had observed that $t_0 = t(\gamma)$. Although we have no basic theory for t_0, (30) satisfies three essential features that the tunnelling time should possess:

(a) as $\gamma \to 0$, $t_0 \to 0$, because tunnelling no longer takes place;
(b) $\gamma \geq t_0$ always because t_0 is an actual tunnelling time while γ is for the fictitious (reference) adiabatic tunnelling time;
(c) t_0 cannot grow as fast as γ because an increase in γ can arise from an increase in the frequency of the field, which in turn makes tunnelling more likely to occur.

5 Conclusion

We have presented an essentially exact formulation of the dynamics of a δ-atom subjected to an oscillating electric field within a time-dependent semiclassical framework (i.e. $\hbar\omega^3/\mu^2 \to 0$). By approximating the phase by its time-average value, we obtained the results of Ergenzinger. We also highlighted the important role played by the imaginary tunnelling time t_0, which is distinct from the adiabatic tunnelling time.

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